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Invariant two-spaces and canonical forms for the Ricci tensor in general relativity

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Abstract. An algebraic classification of the Ricci tensor is given in terms of its invariant two-space structure. The method involves classifying a complex fourth-order tensor which is algebraically equivalent to the trace-free Ricci tensor and which has all the algebraic symmetries of the (complex) Riemann tensor.

1. Introduction

The algebraic classification of the Ricci tensor in general relativity has been discussed by several authors. The first complete solution to the problem appears to have been given by Churchill (1932), who used the invariant two-space structure of the Ricci tensor to develop his classification. Plebanski (1964) gave both a tensorial and a spinorial approach to the problem whilst an alternative discussion in spinors was given by Ludwig and Scanlon (1971). A more recent tensorial approach was pointed out by Hall (1976). Churchill's paper is interesting in that whereas most algebraic discussions of the Ricci tensor tend to be concerned with its eigenvector structure, Churchill considered the invariant two-spaces of this tensor. However, Churchill's method eventually reduces to an eigenvector approach. This procedure will be taken up again here where it will be shown that a complete classification of the Ricci tensor (more precisely the trace-free Ricci tensor) can easily be given entirely in terms of its invariant two-space structure. In fact, in the (Grassmann) manifold of two-dimensional subspaces of the tangent space at some point on the space–time manifold, those two-spaces which are invariant two-spaces of the Ricci tensor occur naturally in orthogonal pairs and much use is made of this fact. The method utilises a 'complexified' version of the trace-free Ricci tensor and this brings about a simplification in much the same way as does the 'complexification' of the Weyl tensor in the well known Petrov classification.

2. The structure of the Ricci tensor

Throughout the paper, M will denote a space–time manifold of signature $+2$ and if $p \in M$, $T_p(M)$ will denote the tangent space to M at p. Thus $T_p(M)$ is a real four-dimensional Lorentzian inner product space. Vectorial classifications of the Ricci tensor are usually based on the fact that the mixed Ricci tensor with components $R_a{}^b$ in

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some chart about p determines a linear map R: $T_p(M) \to T_p(M)$. The general Segré type classification of R then proceeds from the usual Jordan and rational canonical forms for a 4×4 real matrix. (A straightforward way of doing this which avoids use of the rational canonical form is outlined in the appendix.) In this paper, the action of R on the manifold of two-dimensional subspaces of $T_p(M)$ will be considered and R will be classified according to its invariant two-space structure. A two-dimensional subspace (two-space) V of $T_p(M)$ is called an *invariant two-space* of R if whenever $v \in V$, then $R(v) \in V$. The two-dimensional subspaces of $T_p(M)$ are exhaustively characterised according as they contain exactly two, one or no null vectors and are then respectively called time-like, null and space-like. The two-space orthogonal to a space-like (respectively time-like, null) two-space is time-like (respectively space-like, null). The following results concerning the algebraic structure of the Ricci tensor at p can now be stated:

(i) There always exists an invariant two-space of R.
(ii) If V is an invariant two-space of R, then so is the two-space orthogonal to V.
(iii) R has a space-like (equivalently time-like) invariant two-space $\Leftrightarrow R$ has two orthogonal space-like eigenvectors.
(iv) R has a null invariant two-space $\Leftrightarrow R$ has a null eigenvector.
(v) R has at least two distinct orthogonal eigenvectors, at least one of which is space-like.

The result (i) was given by Rainich (1925) whilst the remainder are contained in Churchill's paper (1932). Simpler proofs of all these results were given by Hall (1976).

At this point, it is convenient to set forth the general statement of the Segré type classification of the mapping R. At any point $p \in M$, the Ricci tensor must assume exactly one of the following four canonical types (together with their possible degeneracies): (*a*) R is diagonalisable† (Segré type {1, 1, 1, 1}); (*b*) R has Segré type {2, 1, 1}; (*c*) R has Segré type {3, 1}; (*d*) R has two complex (non-real) and two real eigenvalues, the latter two eigenvalues being associated with simple elementary divisors of R (see the appendix). Concerning these types, one can make the following remarks:

(vi) R has a time-like eigenvector at $p \Leftrightarrow R$ is diagonalisable at p‡.
(vii) If R has two null eigenvectors at p with distinct directions, then R is diagonalisable at p and the corresponding eigenvalues are equal.
(viii) R has all its eigenvalues real but possesses a non-simple elementary divisor at p (that is it belongs to type (*b*) or (*c*) above) $\Leftrightarrow R$ has a unique null eigendirection at p.

These results can all be deduced by inspection of the canonical form for R, but more direct algebraic proofs are available.

3. Invariant two-spaces

From now on, we restrict attention to the trace-free Ricci tensor $\tilde{R}$ where, in components, $\tilde{R}_{ab} = R_{ab} - \frac{1}{4}Rg_{ab}$ with $R = R_{ab}g^{ab}$ the Ricci scalar and g_{ab} the metric tensor. An eigenvector (invariant two-space) of the Ricci tensor is of course an eigenvector (invariant two-space) of the trace-free Ricci tensor and vice versa. Let u,

† The term 'diagonalisable' will always mean 'diagonalisable over $\mathbb{R}$'. In case (*d*), the Ricci tensor is diagonalisable over $\mathbb{C}$ but not over $\mathbb{R}$.

‡ It follows that static space–times always have diagonalisable Ricci tensors since the Ricci tensor always admits a time-like eigenvector (Ehlers and Kundt 1962).

$v \in T_p(M)$ span an invariant two-space V of $\tilde{R}$ at p. Then in components one has

$$u^a\tilde{R}_{ab} = \alpha u_b + \beta v_b \qquad v^a\tilde{R}_{ab} = \gamma u_b + \delta v_b \tag{3.1}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. If $F = u \wedge v$ is the simple bivector at p associated with the two-space V, then it easily follows that the conditions (3.1) are equivalent to

$$\tilde{R}_{b[a}F^b{}_{c]} = \lambda F_{ac} \tag{3.2}$$

where F_{ab} are the covariant components of F at p, where $\lambda \in \mathbb{R}$ and where the square brackets denote the usual antisymmetrisation.

Equation (3.2) can now be written in either of the equivalent forms

$$\text{(i)}\quad E_{abcd}F^{cd} = 2\lambda F_{ab} \qquad \text{(ii)}\quad E_{abcd}\overset{*}{F}{}^{cd} = -2\lambda \overset{*}{F}_{ab} \tag{3.3}$$

$$\overset{+}{E}_{abcd}\overset{+}{F}{}^{cd} = 4\lambda \bar{F}_{ab} \tag{3.4}$$

where

$$\begin{aligned} E_{abcd} &= \tfrac{1}{2}(\tilde{R}_{ac}g_{bd} - \tilde{R}_{bc}g_{ad} + \tilde{R}_{bd}g_{ac} - \tilde{R}_{ad}g_{bc}) \\ \overset{+}{E}_{abcd} &= E_{abcd} + iE^*_{abcd} \qquad F_{ab} = F_{ab} \pm \mathrm{i}\overset{*}{F}_{ab} \end{aligned} \tag{3.5}$$

and where an asterisk in the appropriate place denotes the left or right duality operator. The components E_{abcd} have all the algebraic symmetries of the Riemann tensor components R_{abcd} and are related to them and to the Weyl tensor components C_{abcd} by the equation

$$R_{abcd} = C_{abcd} + E_{abcd} + \tfrac{1}{12}R(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{3.6}$$

Also, the tensor E satisfies the relations

$$\begin{aligned} &-E_{ac}{}^c{}_b = \tilde{R}_{ab} \qquad {}^*E_{abcd} = -E^*_{abcd} \\ &\overset{+}{E}_{abcd} = \overline{\overset{+}{E}_{cdab}} \qquad \overset{+}{E}{}^*_{abcd} = -i\overset{+}{E}_{abcd} \qquad \overset{+}{E}_{ac}{}^c{}_b = E_{ac}{}^c{}_b = -\tilde{R}_{ab}. \end{aligned} \tag{3.7}$$

The conditions of equation (3.3) show that an invariant two-space of $\tilde{R}$ at p determines (to within a scaling factor) a simple 'eigen-bivector' of E and conversely†. The simple eigen-bivectors of E occur in dual pairs just as the invariant two-spaces of $\tilde{R}$ occur in orthogonal pairs. Such a pair of invariant two-spaces then determines (to within a scaling factor) a self-dual–anti-self-dual pair of complex bivectors according to (3.4) and conversely.

Although interest here centres on simple eigen-bivectors of E, this is in a sense the same as considering all eigen-bivectors of E (that is all bivectors F satisfying (3.3(i)) at p) for if this equation holds at p with $\lambda \neq 0$ then F is necessarily simple. If (3.3(i)) holds with $\lambda = 0$ then although F need not be simple at p it determines, to within a scaling factor, a dual pair of simple bivectors at p which satisfy (3.3) (i) with $\lambda = 0$ at p.

Also, if (3.4) holds with $0 \neq \lambda \in \mathbb{C}$, then a duality rotation of F reveals a new self-dual–anti-self-dual pair of complex bivectors satisfying (3.4) with $0 \neq \lambda \in \mathbb{R}$. consequently, each pair of orthogonal invariant two-spaces of $\tilde{R}$ determines (to within a complex factor) a self-dual–anti-self-dual pair of 'complex eigen-bivectors' of $\overset{+}{E}_{abcd}$ satisfying (3.4) with $\lambda \in \mathbb{C}$ and conversely. So the problem of determining the invariant two-spaces of $\tilde{R}$ becomes that of solving the equations (3.4).

It is now convenient to introduce a complex null tetrad $\{l, m, t, \bar{t}\}$ at p with l, m real null vectors and t a complex null vector. The only non-vanishing inner products

† If $\lambda = 0$ and $\overset{+}{F}_{ab}$ is null in (3.4), infinitely many pairs of null invariant two-spaces are admitted.

between these vectors are $l.m = t.\bar{t} = 1$. From this null tetrad (suitably oriented) one can construct a spanning set of three independent complex self-dual bivectors (Sachs 1961)

$$V_{ab} = 2l_{[a}\bar{t}_{b]} \qquad M_{ab} = 2l_{[a}m_{b]} + 2\bar{t}_{[a}t_{b]} \qquad U_{ab} = 2m_{[a}t_{b]}. \tag{3.8}$$

These bivectors and their anti-self-dual conjugates satisfy the completeness relations

$$\begin{aligned} g_{a[c}g_{d]b} + \tfrac{1}{2}i\sqrt{-g}\epsilon_{abcd} &= V_{ab}U_{cd} + U_{ab}V_{cd} - \tfrac{1}{2}M_{ab}M_{cd} \\ g_{a[c}g_{d]b} - \tfrac{1}{2}i\sqrt{-g}\epsilon_{abcd} &= \bar{V}_{ab}\bar{V}_{cd} + \bar{U}_{ab}\bar{V}_{cd} - \tfrac{1}{2}\bar{M}_{ab}\bar{M}_{cd} \end{aligned} \tag{3.9}$$

where ϵ_{abcd} is the alternating symbol: $\epsilon_{abcd} = \epsilon_{[abcd]}$ and $\epsilon_{0123} = 1$.

The discussion presented here can be thought of conveniently in terms of spinors. Each complex eigen-bivector according to (3.4) determines a symmetric eigen-two-spinor of the trace-free Ricci spinor $\Phi_{AB\dot{X}\dot{Y}}$. Indeed, equation (3.4) is equivalent to the spinor relation $\Phi_{AB\dot{X}\dot{Y}}\phi^{AB} = 4\lambda\bar{\phi}_{\dot{X}\dot{Y}}$ where ϕ_{AB} is the symmetric two-spinor associated with the complex self-dual bivector $\overset{+}{F}_{ab}$.

4. The classification

The completeness relations (3.9) and the dual condition on $\overset{+}{E}$ contained in (3.7) allow a decomposition of the tensor $\overset{+}{E}$ at p along the basis members V, M, U and their conjugates. The result is

$$\begin{aligned} \overset{+}{E}_{abcd} = {} & E_1\bar{U}_{ab}U_{cd} + E_2\bar{V}_{ab}V_{cd} + E_3\bar{M}_{ab}M_{cd} + E_4\bar{U}_{ab}V_{cd} + E_5\bar{V}_{ab}U_{cd} \\ & + E_6\bar{U}_{ab}M_{cd} + E_7\bar{M}_{ab}U_{cd} + E_8\bar{V}_{ab}M_{cd} + E_9\bar{M}_{ab}V_{cd} \end{aligned} \tag{4.1}$$

where $E_1, E_2, \ldots, E_9 \in \mathbb{C}$ and where the Hermitian symmetry condition on $\overset{+}{E}$ reveals that $E_1, E_2, E_3 \in \mathbb{R}$ and that $E_5 = \bar{E}_4$, $E_7 = \bar{E}_6$ and $E_9 = \bar{E}_8$. The classification is based on the number of independent self-dual–anti-self-dual pairs of complex bivectors satisfying (3.4) and their type (null or non-null). That at least one such pair exists is guaranteed by the results (i) and (ii) in § 2 and the discussion in § 3.

4.1. Case 1

Suppose $\overset{+}{E}$ admits exactly one independent solution of (3.4) and that it is non-null. Then the relevant solution in (3.4) may be taken as the bivector M and so $\overset{+}{E}$ takes the form (4.1) with $E_6 = E_7 = E_8 = E_9 = 0$. In order that U and V should not also satisfy (3.4) one requires $E_1 \neq 0$ and $E_2 \neq 0$. A null rotation of the complex null tetrad may then be performed which preserves the above conditions but which gives $|E_1| = |E_2|$. If $E_1 = E_2$, the extra (non-null) solutions $U \pm V$ are also admitted. This case will be dealt with later, the present case requiring $E_1 = -E_2$. The resulting form for the tensor $\overset{+}{E}$ then yields $\tilde{R}$ from (3.7) and so leads to the Ricci tensor

$$R_{ab} = \rho_1(l_al_b - m_am_b) + 2\rho_2 l_{(a}m_{b)} + \rho_3 x_a x_b + \rho_4 y_a y_b \tag{4.2}$$

at p, where $\rho_1, \rho_2, \rho_3, \rho_4 \in \mathbb{R}$ with $\rho_1 \neq 0$ and where $t_a\sqrt{2} = x_a + iy_a$ with $x^ax_a = y^ay_a = 1$. Here the Ricci tensor has two non-real and two real eigenvalues (see the appendix). It is readily shown that no other independent solutions of (3.4) exist.

4.2. Case 2

Suppose now that $\overset{+}{E}$ admits exactly one independent solution of (3.4) and that it is null. Then the relevant solution of (3.4) may be taken as the bivector V. This condition together with the condition preventing the bivector M from also becoming a solution leads to $E_1 = E_6 = E_7 = 0$, $E_8 \neq 0$, a null rotation having been used to make E_8 real. The condition that no other independent solutions of (3.4) exist is then $E_4 + 2E_3 = 0$. A final null rotation may then be used to set $E_2 = 0$ and the Ricci tensor takes the form

$$R_{ab} = 2\sigma_1 l_{(a}m_{b)} + 2\sigma l_{(a}x_{b)} + \sigma_1 x_a x_b + \sigma_2 y_a y_b \tag{4.3}$$

with σ, σ_1, $\sigma_2 \in \mathbb{R}$ and $\sigma \neq 0$. In fact a null rotation can be used to set $\sigma = 1$. The Ricci tensor (4.3) has Segré type $\{3, 1\}$ or some degeneracy of this type (see the appendix).

4.3. Case 3

Suppose now that $\overset{+}{E}$ admits exactly two independent solutions of (3.4) (together with possibly their linear combinations). It is straightforward to show that in this case there must be a unique null solution of (3.4) together with a non-null solution and that these solutions may be chosen as the bivectors V and M. These conditions lead to $E_1 = E_6 = E_7 = E_8 = E_9 = 0$ and $E_2 \neq 0$ whilst a null rotation may be used to make E_4 real. The Ricci tensor is

$$R_{ab} = 2\alpha_1 l_{(a}m_{b)} + \alpha l_a l_b + \alpha_2 x_a x_b + \alpha_3 y_a y_b \tag{4.4}$$

where α, α_1, α_2, $\alpha_3 \in \mathbb{R}$ with $\alpha \neq 0$. If $\alpha > 0$ ($\alpha < 0$), a null rotation can be used to set $\alpha = 1$ ($\alpha = -1$). The Ricci tensor (4.4) has Segré type $\{2, 1, 1\}$ or some degeneracy of this type (see the appendix).

4.4. Case 4

Finally suppose $\overset{+}{E}$ admits three independent solutions of (3.4) (together with possibly their linear combinations). It easily follows that the number of independent null solutions is either zero or at least two. In the former case one is lead to case 1 with $E_1 = E_2$ and so

$$R_{ab} = \rho_1(l_a l_b + m_a m_b) + 2\rho_2 l_{(a}m_{b)} + \rho_3 x_a x_b + \rho_4 y_a y_b \tag{4.5}$$

where $\rho_1, \rho_2, \rho_3, \rho_4 \in \mathbb{R}$, $\rho_1 \neq 0$ and where the extra conditions $\rho_3 \neq \rho_2 - \rho_1 \neq \rho_4$ hold. The Ricci tensor (4.5) has Segré type $\{1, 1, 1, 1\}$ or some degeneracy of this type (see the appendix). In the latter case one may arrange the null tetrad so that the null bivectors concerned are U and V. The bivector M then becomes a third independent solution of (3.4) and so $E_1 = E_2 = E_6 = E_7 = E_8 = E_9 = 0$ whilst a null rotation may be used to make E_4 real. Then

$$R_{ab} = 2\beta_1 l_{(a}m_{b)} + \beta_2 x_a x_b + \beta_3 y_a y_b \tag{4.6}$$

where β_1, β_2, $\beta_3 \in \mathbb{R}$. The Ricci tensor (4.6) has Segré type $\{(11), 1, 1\}$ or some degeneracy of this type (see the appendix).

These results may be summarised as follows:

(1) The Ricci tensor has a unique pair of (orthogonal) invariant two-spaces *and* they are *non-null* if and only if it has non-real eigenvalues.

(2) The Ricci tensor has a unique pair of (orthogonal) invariant two-spaces *and* they are *null* if and only if it has Segré type $\{3, 1\}$. In the degenerate case $\{(3, 1)\}$, infinitely many pairs of null invariant two-spaces occur, all based on the null eigenvector l.

(3) The Ricci tensor has exactly two independent pairs of invariant two-spaces if and only if it has Segré type $\{2, 1, 1\}$ or one of its degeneracies. Exactly one of these pairs of invariant two-spaces is null if $\alpha_2 \neq \alpha_3$ in (4.4). If $\alpha_2 = \alpha_3$ then infinitely many pairs of null invariant two-spaces occur, all based on the null eigenvector l.

(4) The Ricci tensor has exactly three independent pairs of invariant two-spaces if and only if it is diagonalisable (Segré type $\{1, 1, 1, 1\}$ or some degeneracy of this type). The number of pairs of null invariant two-spaces is either zero or at least two, and in the latter case, null invariant two-spaces with different null directions exist.

5. The Bel criteria

In the classification of the Weyl tensor, the well known Bel criteria (Bel 1962) give a convenient description of the Petrov classification at $p \in M$ in terms of certain contracted relationships between the Weyl tensor and its associated null directions at p. Criteria similar to those of Bel can be given for the tensor $E(\neq 0)$ at p and the following relationships between such criteria and the Ricci tensor type at p can be proved:

(*a*) The *trace-free* Ricci tensor has Segré type $\{2, 1, 1\}$ with all eigenvalues zero if and only if there exists $l \in T_p(M)$, $l \neq 0$ such that $l^a E_{abcd} = 0$. The vector l is necessarily null, unique up to a scaling factor and coincides with the (unique) null Ricci eigenvector.

(*b*) The *trace-free* Ricci tensor has Segré type $\{3, 1\}$ with all eigenvalues zero if and only if there exists a non-zero null bivector F at p and a vector $l \in T_p(M)$, $l \neq 0$ such that $l^a E_{abcd} = l_b F_{cd}$. The vector l is again necessarily null, unique up to a scaling factor and coincides with the (unique) null Ricci eigenvector and the repeated principal null direction of F.

(*c*) A null vector $l \in T_p(M)$, $l \neq 0$ is a Ricci eigenvector if and only if there exists $\alpha \in \mathbb{R}$ such that $l^a l^c E_{abcd} = a l_b l_d$.

In (*a*), alternative equivalent statements are $l^a E^{*}_{abcd} = 0$ and $l^a \overset{+}{E}_{abcd} = 0$ and there is a similar obvious alternative for (*b*). In (*c*), alternative equivalent statements are $l^a l^c E^{*}_{abcd} = 0$ and $l^a l^c \overset{+}{E}_{abcd} = \alpha l_b l_d$ ($\alpha \in \mathbb{R}$) for some null vector l. It is remarked that infinitely many distinct null directions may satisfy the conditions of (*c*) above (unlike the equivalent case for the Weyl tensor). The existence of at least two distinct null directions satisfying condition (*c*) is equivalent to the Ricci tensor being diagonalisable and possessing precisely those null directions as eigendirections. The equivalent of the Debever–Penrose condition on the Weyl tensor (Sachs 1961) may not be investigated on the tensor E.

(*d*) If l is a null vector in $T_p(M)$, $l \neq 0$, then $R_{ab} l^a l^b = 0$ if and only if $l^b l^c l_{[e} E_{a]bc[d} l_{f]} = 0$.

In (*d*), equivalent conditions are $l^b l^c l_{[e} E^{*}_{a]bc[d} l_{f]} = 0$ and $l^b l^c l_{[e} \overset{+}{E}_{a]bc[d} l_{f]} = 0$ for some null vector l. Unlike the situation for the Weyl tensor, there may be no null directions or infinitely many distinct null directions satisfying the conditions (*d*). Examples with the latter property are easily constructed, whilst an example of the former property is given by (4·5) with $\rho_1 = \rho_3 = \rho_4 = 1$ $\rho_2 = 0$. The proofs of (a), (b), (c) and (d) can be easily gathered from the result (iv) of § 2 and equations (3.5) and (4.1).

6. Concluding remarks

The preceeding classification applies quite generally to any real symmetric second-order tensor on a four-dimensional Loretzian manifold M. However, the algebraic types apply to such tensors at a point $p \in M$ and for such a tensor in some open set $U \subseteq M$, the algebraic type may not be the same throughout U. If, however, the tensor has non real eigenvalues at p, then it will maintain this feature throughout some neighbourhood of p.

Equation (4.1) when contracted and combined with the obvious relation in (3.7) shows that any real symmetric second-order trace-free tensor at $p \in M$ can be written in terms of at most three (simple) bivectors and their duals, thus generalising the case of the (trace-free) Maxwell energy momentum tensor which can be expressed in terms of a single (simple) bivector and its dual (cf Plebanski 1964).

Of the four algebraic types of tensors discussed here, members of two of them (those with non-real eigenvalues and those with Segré type {3, 1} or some degeneracy of this type) fail to satisfy either of the following two conditions (cf Plebanski 1964, Collinson and Shaw 1972, Hall 1976): for every time-like vector $u \in T_p(M)$: (i) $T_{ab}u^a u^b \geqslant 0$; (ii) $T^{ab}u_b$ is not space-like, where T_{ab} are the components of the symmetric tensor under consideration. Condition (i) is the *weak energy condition* and (i) and (ii) together constitute the *dominant energy condition* (Hawking and Ellis 1973). They place what appear to be physically realistic conditions on the local energy density and energy flow, and ensure that physically reasonable energy–momentum tensors in general relativity must be drawn from (certain subsets of) the other two algebraic types.

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Appendix

In this appendix, a strightforward calculation is presented of the various Segré types for the Ricci tensor at $p \in M$. One seeks solutions $k^a \in T_p(M)$ and $\lambda \in \mathbb{R}$ of the eigenvalue problem $(R_a{}^b - \lambda\delta_a{}^b)k^a = 0$ where the index b is raised so as to convert the problem into the standard form for matrices. The possible canonical forms for $R_a{}^b$ can now be listed. Firstly, there is the case when non-real eigenvalues occur. Secondly, when the eigenvalues are real, one has the following Jordan forms given in terms of their Segré types: (i) {1, 1, 1, 1} (diagonalisable case); (ii) {2, 1, 1}; (iii) {2, 2}; (iv) {3, 1}; (v) {4}. Because of the Lorentz signature of the metric tensor, the cases {2, 2} and {4} cannot occur. For example suppose $R_a{}^b$ has Segré type {4} at $p \in M$. Then there exists a basis of $T_p(M)$ in which $R_a{}^b$ takes the canonical Jordan form for this Segré type. The condition that $R_a{}^b g_{bc}$ be symmetric then implies certain algebraic constraints on the components of g_{ab} in the above basis from which one readily deduces that $\det(g_{ab}) > 0$. This contradicts the Lorentz signature of g_{ab}. A similar argument removes the possibility of the Segré type {2, 2} occuring (where because of results (vii) and (viii) of § 2 and the fact that eigenvectors corresponding to distinct eigenvalues are orthogonal, one need only consider the case when the eigenvalues are equal).

In the other cases when real eigenvalues exist, one can write down the canonical Jordan form in an appropriate basis in each case and use the symmetry of $R_a{}^b g_{bc}$ to impose conditions on the components g_{ab} in that basis. The Jordan form then allows one to write out a 'canonical' form for $R_a{}^b$ in terms of certain vectors, the constraints imposed on g_{ab} then providing the orthogonality conditions on these vectors. For the Segré types {1, 1, 1, 1}, {2, 1, 1} and {3, 1}, one is easily led to the general expressions (4.5), (4.4) and (4.3) respectively, where now the only condition on these equations are $\alpha \neq 0$ and $\sigma \neq 0$.

In the case when complex eigenvalues occur, one can evaluate the canonical form easily by the following considerations. If $R_a{}^b$ has the complex eigenvectors $s \pm it$ ($s, t \in T_p(M)$) with corresponding eigenvalues $a \pm ib$ ($a, b \in \mathbb{R}$, $b \neq 0$), then the fact that these eigenvalues are different implies that $s \pm it$ are orthogonal. Hence $s^a s_a + t^a t_a = 0$. Also, one may assume that $s^a t_a = 0$, for if not, then there exists a non-zero complex multiple of $s \pm it$ whose real and imaginary parts are orthogonal. (The possibility that s and t are parallel and null is easily ruled out since this would imply that $b = 0$.) Hence one of the real vectors s and t is time-like and the other is space-like and it easily follows that they span a (time-like) invariant two-space of R at p. The result (iii) of § 2 then shows the existence of two (real) space-like eigenvectors of R and consequently R is diagonalisable over $\mathbb{C}$. The obvious diagonal form for R then leads to equation (4.2) where real null vectors l and m, proportional to $s \pm t$, are introduced.

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